

RELAXATION OF p -GROWTH INTEGRAL FUNCTIONALS UNDER SPACE-DEPENDENT DIFFERENTIAL CONSTRAINTS

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ABSTRACT. A representation formula for the relaxation of integral energies

$$(u, v) \mapsto \int_{\Omega} f(x, u(x), v(x)) \, dx,$$

is obtained, where f satisfies p -growth assumptions, $1 < p < +\infty$, and the fields v are subjected to space-dependent first order linear differential constraints in the framework of \mathcal{A} -quasiconvexity with variable coefficients.

1. INTRODUCTION

The analysis of constrained relaxation problems is a central question in materials science. Many applications in continuum mechanics and, in particular, in magnetoelasticity, rely on the characterization of minimizers of non-convex multiple integrals of the type

$$u \mapsto \int_{\Omega} f(x, u(x), \nabla u(x), \dots, \nabla^k u(x)) \, dx$$

or

$$(u, v) \mapsto \int_{\Omega} f(x, u(x), v(x)) \, dx, \quad (1.1)$$

where Ω is an open, bounded subset of \mathbb{R}^N , $u : \Omega \rightarrow \mathbb{R}^m$, $m \in \mathbb{N}$, and the fields $v : \Omega \rightarrow \mathbb{R}^d$, $d \in \mathbb{N}$, satisfy partial differential constraints of the type “ $\mathcal{A}v = 0$ ” other than $\operatorname{curl} v = 0$ (see e.g. [5, 9]).

In this paper we provide a representation formula for the relaxation of non-convex integral energies of the form (1.1), in the case in which the energy density f satisfies p -growth assumptions, and the fields v are subjected to linear first-order space-dependent differential constraints.

The natural framework to study this family of relaxation problems is within the theory of \mathcal{A} -quasiconvexity with variable coefficients. In order to present this notion, we need to introduce some notation.

For $i = 1 \dots, N$, let $A^i \in C^\infty(\mathbb{R}^N; \mathbb{M}^{l \times d}) \cap W^{1,\infty}(\mathbb{R}^N; \mathbb{M}^{l \times d})$, let $1 < p < +\infty$, and consider the differential operator

$$\mathcal{A} : L^p(\Omega; \mathbb{R}^d) \rightarrow W^{-1,p}(\Omega; \mathbb{R}^l), \quad d, l \in \mathbb{N},$$

defined as

$$\mathcal{A}v := \sum_{i=1}^N A^i(x) \frac{\partial v(x)}{\partial x_i} \quad (1.2)$$

for every $v \in L^p(\Omega; \mathbb{R}^d)$, where (1.2) is to be interpreted in the sense of distributions. Assume that the symbol $\mathbb{A} : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{M}^{l \times d}$,

$$\mathbb{A}(x, w) := \sum_{i=1}^N A^i(x) w_i \quad \text{for } (x, w) \in \mathbb{R}^N \times \mathbb{R}^N,$$

2010 *Mathematics Subject Classification.* 49J45; 35D99; 49K20.

Key words and phrases. Relaxation, \mathcal{A} -quasiconvexity.

satisfies the uniform constant rank condition (see [22])

$$\text{rank } \mathbb{A}(x, w) = r \quad \text{for every } x \in \mathbb{R}^N \text{ and } w \in \mathbb{S}^{n-1}. \quad (1.3)$$

Let Q be the unit cube in \mathbb{R}^N with sides parallel to the coordinate axis, i.e.,

$$Q := \left(-\frac{1}{2}, \frac{1}{2} \right).$$

Denote by $C_{\text{per}}^\infty(\mathbb{R}^N; \mathbb{R}^m)$ the set of \mathbb{R}^m -valued smooth maps that are Q -periodic in \mathbb{R}^N , and for every $x \in \Omega$ consider the set

$$\mathcal{C}_x := \left\{ w \in C_{\text{per}}^\infty(\mathbb{R}^N; \mathbb{R}^m) : \int_Q w(y) dy = 0, \text{ and } \sum_{i=1}^N A^i(x) \frac{\partial w(y)}{\partial y_i} = 0 \right\}.$$

Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \rightarrow [0, +\infty)$ be a Carathéodory function. The \mathcal{A} -quasiconvex envelope of $f(x, u, \cdot)$ for $x \in \Omega$ and $u \in \mathbb{R}^m$ is defined for $\xi \in \mathbb{R}^d$ as

$$Q_{\mathcal{A}(x)} f(x, u, \xi) := \inf \left\{ \int_Q f(x, u, \xi + w(y)) dy : w \in \mathcal{C}_x \right\}.$$

We say that f is \mathcal{A} -quasiconvex if $f(x, u, \xi) = Q_{\mathcal{A}(x)} f(x, u, \xi)$ for a.e. $x \in \Omega$, and for all $u \in \mathbb{R}^m$ and $\xi \in \mathbb{R}^d$.

The notion of \mathcal{A} -quasiconvexity was first introduced by B. Dacorogna in [8], and extensively characterized in [17] by I. Fonseca and S. Müller for operators \mathcal{A} defined as in (1.2), satisfying the constant rank condition (1.3), and having constant coefficients,

$$A^i(x) \equiv A^i \in \mathbb{M}^{l \times d} \quad \text{for every } x \in \mathbb{R}^N, i = 1, \dots, N.$$

In that paper the authors proved (see [17, Theorems 3.6 and 3.7]) that under p -growth assumptions on the energy density f , \mathcal{A} -quasiconvexity is necessary and sufficient for the lower-semicontinuity of integral functionals

$$I(u, v) := \int_\Omega f(x, u(x), v(x)) dx \quad \text{for every } (u, v) \in L^p(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^d)$$

along sequences (u^n, v^n) satisfying $u^n \rightarrow u$ in measure, $v^n \rightharpoonup v$ in $L^p(\Omega; \mathbb{R}^d)$, and $\mathcal{A}v^n \rightarrow 0$ in $W^{-1,p}(\Omega)$. We remark that in the framework $\mathcal{A} = \text{curl}$, i.e., when $v^n = \nabla \phi^n$ for some $\phi^n \in W^{1,p}(\Omega; \mathbb{R}^m)$, $d = n \times m$, \mathcal{A} -quasiconvexity reduces to Morrey's notion of quasiconvexity.

The analysis of properties of \mathcal{A} -quasiconvexity for operators with constant coefficients was extended in the subsequent paper [6], where A. Braides, I. Fonseca and G. Leoni provided an integral representation formula for relaxation problems under p -growth assumptions on the energy density, and presented (via Γ -convergence) homogenization results for periodic integrands evaluated along \mathcal{A} -free fields. These homogenization results were later generalized in [13], where I. Fonseca and S. Krömer worked under weaker assumptions on the energy density f . In [19, 20], simultaneous homogenization and dimension reduction was studied in the framework of \mathcal{A} -quasiconvexity with constant coefficients. Oscillations and concentrations generated by \mathcal{A} -free mappings are the subject of [14]. Very recently an analysis of the case in which the energy density is nonpositive has been carried out in [18], and applications to the theory of compressible Euler systems have been studied in [7]. A parallel analysis for operators with constant coefficients and under linear growth assumptions for the energy density has been developed in [1, 4, 15, 21]. A very general characterization in this setting has been obtained in [2], following the new insight in [12].

The theory of \mathcal{A} -quasiconvexity for operators with variable coefficients has been characterized by P. Santos in [23]. Homogenization results in this setting have been obtained in [10] and [11].

This paper is devoted to proving a representation result for the relaxation of integral energies in the framework of \mathcal{A} -quasiconvexity with variable coefficients. To be precise, let $1 < p, q < +\infty$, $d, m, l \in \mathbb{N}$, and consider a Carathéodory function $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \rightarrow [0, +\infty)$ satisfying

$$(H) \quad 0 \leq f(x, u, v) \leq C(1 + |u|^p + |v|^q), \quad 1 < p, q < +\infty,$$

for a.e. $x \in \Omega$, and all $(u, v) \in \mathbb{R}^m \times \mathbb{R}^d$, with $C > 0$.

Denoting by $\mathcal{O}(\Omega)$ the collection of open subsets of Ω , for every $D \in \mathcal{O}(\Omega)$, $u \in L^p(\Omega; \mathbb{R}^m)$ and $v \in L^q(\Omega; \mathbb{R}^d)$ with $\mathcal{A}v = 0$, we define

$$\mathcal{I}((u, v), D) := \inf \left\{ \liminf_{n \rightarrow +\infty} \int_D f(x, u_n(x), v_n(x)) : u_n \rightarrow u \text{ strongly in } L^p(\Omega; \mathbb{R}^m), \right. \\ \left. v_n \rightharpoonup v \text{ weakly in } L^q(\Omega; \mathbb{R}^d) \text{ and } \mathcal{A}v_n \rightarrow 0 \text{ strongly in } W^{-1,q}(\Omega; \mathbb{R}^l) \right\}. \quad (1.4)$$

Our main result is the following.

Theorem 1.1. *Let \mathcal{A} be a first order differential operator with variable coefficients, satisfying (1.3). Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \rightarrow [0, +\infty)$ be a Carathéodory function satisfying (H). Then,*

$$\int_D Q_{\mathcal{A}(x)} f(x, u(x), v(x)) dx = \mathcal{I}((u, v), D)$$

for all $D \in \mathcal{O}(\Omega)$, $u \in L^p(\Omega; \mathbb{R}^m)$ and $v \in L^q(\Omega; \mathbb{R}^d)$ with $\mathcal{A}v = 0$.

Adopting the “blow-up” method introduced in [16], the proof of the theorem consists in showing that the functional $\mathcal{I}((u, v), \cdot)$ is the trace of a Radon measure absolutely continuous with respect to the restriction of the Lebesgue measure \mathcal{L}^N to Ω , and proving that for a.e. $x \in \Omega$ the Radon-Nicodym derivative $\frac{d\mathcal{I}((u, v), \cdot)(x)}{d\mathcal{L}^N}$ coincides with the \mathcal{A} -quasiconvex envelope of f .

The arguments used are a combination of the ideas from [6, Theorem 1.1] and from [23]. The main difference with [6, Theorem 1.1], which reduces to our setting in the case in which the operator \mathcal{A} has constant coefficients, is in the fact that while defining the operator \mathcal{I} in (1.4) we can not work with exact solutions of the PDE, but instead we need to study sequences of asymptotically \mathcal{A} -vanishing fields. As pointed out in [23], in the case of variable coefficients the natural framework is the context of pseudo-differential operators. In this setting, we don’t know how to project directly onto the kernel of the differential constraint, but we are able to construct an “approximate” projection operator P such that for every field $v \in L^p$, the $W^{-1,p}$ norm of $\mathcal{A}Pv$ is controlled by the $W^{-1,p}$ norm of v itself (we refer to [23, Subsection 2.1] for a detailed explanation of this issue and to the references therein for a treatment of the main properties of pseudo-differential operators). For the same reason, in the proof of the inequality

$$\frac{d\mathcal{I}((u, v), \cdot)(x)}{d\mathcal{L}^N} \leq Q_{\mathcal{A}(x)} f(x, u(x), v(x)) \quad \text{for a.e. } x \in \Omega,$$

an equi-integrability argument is needed (see Proposition 3.2). We also point out that the representation formula in Theorem 1.1 was obtained in a simplified setting in [11] as a corollary of the main homogenization result. Here we provide an alternative, direct proof, which does not rely on homogenization techniques.

The paper is organized as follows: in Section 2 we establish the main assumptions on the differential operator \mathcal{A} and we recall some preliminary results on \mathcal{A} -quasiconvexity with variable coefficients. Section 3 is devoted to the proof of Theorem 1.1.

Notation

Throughout the paper $\Omega \subset \mathbb{R}^N$ is a bounded open set, $1 < p, q < +\infty$, $\mathcal{O}(\Omega)$ is the set of open subsets of Ω , Q denotes the unit cube in \mathbb{R}^N , $Q(x_0, r)$ and $B(x_0, r)$ are, respectively, the open cube and the

open ball in \mathbb{R}^N , with center x_0 and radius r . Given an exponent $1 < q < +\infty$, we denote by q' its conjugate exponent, i.e., $q' \in (1, +\infty)$ is such that

$$\frac{1}{q} + \frac{1}{q'} = 1.$$

Whenever a map $v \in L^q, C^\infty, \dots$ is Q -periodic, that is

$$v(x + e_i) = v(x) \quad i = 1, \dots, N,$$

for a.e. $x \in \mathbb{R}^N$, $\{e_1, \dots, e_N\}$ being the standard basis of \mathbb{R}^N , we write $v \in L^q_{\text{per}}, C^\infty_{\text{per}}, \dots$. We implicitly identify the spaces $L^q(Q)$ and $L^q_{\text{per}}(\mathbb{R}^N)$.

We adopt the convention that C will denote a generic constant, whose value may change from line to line in the same formula.

2. PRELIMINARY RESULTS

In this section we introduce the main assumptions on the differential operator \mathcal{A} and we recall some preliminary results about \mathcal{A} -quasiconvexity.

For $i = 1, \dots, N$, $x \in \mathbb{R}^N$, consider the linear operators $A^i(x) \in \mathbb{M}^{l \times d}$, with $A^i \in C^\infty(\mathbb{R}^N; \mathbb{M}^{l \times d}) \cap W^{1,\infty}(\mathbb{R}^N; \mathbb{M}^{l \times d})$. For every $v \in L^q(\Omega; \mathbb{R}^d)$ we set

$$\mathcal{A}v := \sum_{i=1}^N A^i(x) \frac{\partial v(x)}{\partial x_i} \in W^{-1,q}(\Omega; \mathbb{R}^l).$$

The symbol $\mathbb{A} : \mathbb{R}^N \times \mathbb{R}^N \setminus \{0\} \rightarrow \mathbb{M}^{l \times d}$ associated to the differential operator \mathcal{A} is

$$\mathbb{A}(x, \lambda) := \sum_{i=1}^N A^i(x) \lambda_i \in \mathbb{M}^{l \times d}$$

for every $x \in \mathbb{R}^N$, $\lambda \in \mathbb{R}^N \setminus \{0\}$. We assume that \mathcal{A} satisfies the following *uniform constant rank condition*:

$$\text{rank} \left(\sum_{i=1}^N A^i(x) \lambda_i \right) = r \quad \text{for all } x \in \mathbb{R}^N \text{ and } \lambda \in \mathbb{R}^N \setminus \{0\}. \quad (2.1)$$

For every $x \in \mathbb{R}^N$, $\lambda \in \mathbb{R}^N \setminus \{0\}$, let $\mathbb{P}(x, \lambda) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be the linear projection on $\text{Ker } \mathbb{A}(x, \lambda)$, and let $\mathbb{Q}(x, \lambda) : \mathbb{R}^l \rightarrow \mathbb{R}^d$ be the linear operator given by

$$\begin{aligned} \mathbb{Q}(x, \lambda) \mathbb{A}(x, \lambda) v &:= v - \mathbb{P}(x, \lambda) v \quad \text{for all } v \in \mathbb{R}^d, \\ \mathbb{Q}(x, \lambda) \xi &= 0 \quad \text{if } \xi \notin \text{Range } \mathbb{A}(x, \lambda). \end{aligned}$$

The main properties of $\mathbb{P}(\cdot, \cdot)$ and $\mathbb{Q}(\cdot, \cdot)$ are recalled in the following proposition (see *e.g.* [23, Subsection 2.1]).

Proposition 2.1. *Under the constant rank condition (2.1), for every $x \in \mathbb{R}^N$ the operators $\mathbb{P}(x, \cdot)$ and $\mathbb{Q}(x, \cdot)$ are, respectively, 0-homogeneous and (-1) -homogeneous. In addition, $\mathbb{P} \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N \setminus \{0\}; \mathbb{M}^{d \times d})$ and $\mathbb{Q} \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N \setminus \{0\}; \mathbb{M}^{d \times l})$.*

Let $\eta \in C_c^\infty(\Omega; [0, 1])$, $\eta = 1$ in Ω' for some $\Omega' \subset \subset \Omega$. We denote by \mathbb{A}_η the symbol

$$\mathbb{A}_\eta(x, \lambda) := \sum_{i=1}^N \eta(x) A^i(x) \lambda_i, \quad (2.2)$$

for every $x \in \mathbb{R}^N$, $\lambda \in \mathbb{R}^N \setminus \{0\}$, and by \mathcal{A}_η the corresponding pseudo-differential operator (see [23, Subsection 2.1] for an overview of the main properties of pseudo-differential operators). Let $\chi \in C^\infty(\mathbb{R}^+; \mathbb{R})$ be such that $\chi(|\lambda|) = 0$ for $|\lambda| < 1$ and $\chi(|\lambda|) = 1$ for $|\lambda| > 2$. Let also P_η be the operator associated to the symbol

$$\mathbb{P}_\eta(x, \lambda) := \eta^2(x) \mathbb{P}(x, \lambda) \chi(|\lambda|) \quad (2.3)$$

for every $x \in \mathbb{R}^N$, $\lambda \in \mathbb{R}^N \setminus \{0\}$. The following proposition (see [23, Theorem 2.2 and Subsection 2.1]) collects the main properties of the operators P_η and \mathcal{A}_η .

Proposition 2.2. *Let $1 < q < +\infty$, and let \mathcal{A}_η and P_η be the pseudo-differential operators associated with the symbols (2.2) and (2.3), respectively. Then there exists a constant C such that*

$$\|P_\eta v\|_{L^q(\Omega; \mathbb{R}^d)} \leq C \|v\|_{L^q(\Omega; \mathbb{R}^d)} \quad (2.4)$$

for every $v \in L^q(\Omega; \mathbb{R}^d)$, and

$$\begin{aligned} \|P_\eta v\|_{W^{-1,q}(\Omega; \mathbb{R}^d)} &\leq C \|v\|_{W^{-1,q}(\Omega; \mathbb{R}^d)}, \\ \|v - P_\eta v\|_{L^q(\Omega; \mathbb{R}^d)} &\leq C (\|\mathcal{A}_\eta v\|_{W^{-1,q}(\Omega; \mathbb{R}^d)} + \|v\|_{W^{-1,q}(\Omega; \mathbb{R}^d)}), \\ \|\mathcal{A}_\eta P_\eta v\|_{W^{-1,q}(\Omega; \mathbb{R}^d)} &\leq C \|v\|_{W^{-1,q}(\Omega; \mathbb{R}^d)} \end{aligned}$$

for every $v \in W^{-1,q}(\Omega; \mathbb{R}^d)$.

3. PROOF OF THEOREM 1.1

Before proving Theorem 1.1 we state and prove a decomposition lemma, which generalizes [17, Lemma 2.15] to the case of operators with variable coefficients.

Lemma 3.1. *Let $1 < q < +\infty$. Let \mathcal{A} be a first order differential operator with variable coefficients, satisfying (2.1). Let $v \in L^q(\Omega; \mathbb{R}^d)$, and let $\{v_n\}$ be a bounded sequence in $L^q(\Omega; \mathbb{R}^d)$ such that*

$$\begin{aligned} v_n &\rightharpoonup v \quad \text{weakly in } L^q(\Omega; \mathbb{R}^d), \\ \mathcal{A}v_n &\rightarrow 0 \quad \text{strongly in } W^{-1,q}(\Omega; \mathbb{R}^d), \\ \{v_n\} &\text{ generates the Young measure } \nu. \end{aligned}$$

Then, there exists a q -equiintegrable sequence $\{\tilde{v}_n\} \subset L^q(\Omega; \mathbb{R}^d)$ such that

$$\mathcal{A}\tilde{v}_n \rightarrow 0 \quad \text{strongly in } W^{-1,s}(\Omega; \mathbb{R}^d) \quad \text{for every } 1 < s < q, \quad (3.1)$$

$$\begin{aligned} \int_\Omega \tilde{v}_n(x) dx &= \int_\Omega v(x) dx, \\ \tilde{v}_n - v_n &\rightarrow 0 \quad \text{strongly in } L^s(\Omega; \mathbb{R}^d) \quad \text{for every } 1 < s < q, \end{aligned} \quad (3.2)$$

$$\tilde{v}_n \rightharpoonup v \quad \text{weakly in } L^q(\Omega; \mathbb{R}^d). \quad (3.3)$$

In addition, if $\Omega \subset Q$ then we can construct the sequence $\{\tilde{v}_n\}$ so that $\tilde{v}_n - v \in L^q_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d)$ for every $n \in \mathbb{N}$.

Proof. Arguing as in the first part of [23, Proof of Theorem 1.1], we construct a q -equiintegrable sequence $\{\tilde{v}_n\}$ satisfying (3.1), (3.2) and (3.3). The conclusion follows by setting $\tilde{v}_n := \hat{v}_n - \int_\Omega \hat{v}_n(x) dx + \int_\Omega v(x) dx$.

In the case in which $\Omega \subset Q$, let $\{\varphi^i\}$ be a sequence of cut-off functions in Q with $0 \leq \varphi^i \leq 1$ in Q , such that $\varphi^i = 0$ on $Q \setminus \Omega$ and $\varphi^i \rightarrow 1$ pointwise in Ω . Define $w_n^i := \varphi^i(\hat{v}_n - v)$. By (3.3) for every $\psi \in L^{q'}(\Omega; \mathbb{R}^d)$ we have

$$\lim_{i \rightarrow +\infty} \lim_{n \rightarrow +\infty} \int_\Omega w_n^i(x) \psi(x) dx = 0.$$

By (3.1), (3.2), and the compact embedding of $L^q(\Omega; \mathbb{R}^d)$ into $W^{-1,q}(\Omega; \mathbb{R}^d)$, there holds

$$\mathcal{A}w_n^i = \varphi^i \mathcal{A}\hat{v}_n + \left(\sum_{j=1}^N A_j \frac{\partial \varphi^i}{\partial x_j} \right) \hat{v}_n \rightarrow 0 \quad \text{strongly in } W^{-1,s}(\Omega; \mathbb{R}^d)$$

as $n \rightarrow +\infty$, for every $1 < s < q$. Extending the maps w_n^i outside Q by periodicity, by the metrizable topology on bounded sets and by Attouch's diagonalization lemma (see [3, Lemma 1.15 and Corollary 1.16]), we obtain a sequence

$$w_n := w_n^{i(n)},$$

with $\{w_n\} \subset L^q_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d)$, and such that $w_n + v$ satisfies (3.1), (3.2) and (3.3). The thesis follows by setting

$$\tilde{v}_n := w_n - \int_{\Omega} w_n(x) dx + v.$$

□

The following proposition will allow us to neglect vanishing perturbations of q -equiintegrable sequences.

Proposition 3.2. *For every $n \in \mathbb{N}$, let $f_n : Q \times \mathbb{R}^d \rightarrow [0, +\infty)$ be a continuous function. Assume that there exists a constant $C > 0$ such that, for $q > 1$,*

$$\sup_{n \in \mathbb{N}} f_n(y, \xi) \leq C(1 + |\xi|^q) \quad \text{for every } y \in Q \text{ and } \xi \in \mathbb{R}^d, \quad (3.4)$$

and that the sequence $\{f_n(y, \cdot)\}$ is equicontinuous in \mathbb{R}^d , uniformly in y . Let $\{w_n\}$ be a q -equiintegrable sequence in $L^q(Q; \mathbb{R}^d)$, and let $\{v_n\} \subset L^q(Q; \mathbb{R}^d)$ be such that

$$v_n \rightarrow 0 \quad \text{strongly in } L^q(Q; \mathbb{R}^d). \quad (3.5)$$

Then

$$\lim_{n \rightarrow +\infty} \left| \int_Q f_n(y, w_n(y)) dy - \int_Q f_n(y, v_n(y) + w_n(y)) dy \right| = 0.$$

Proof. Fix $\eta > 0$. In view of (3.5), the sequence $\{C(1 + |v_n|^q + |w_n|^q)\}$ is equiintegrable in Q , thus there exists $0 < \varepsilon < \frac{\eta}{3}$ such that

$$\sup_{n \in \mathbb{N}} \int_A C(1 + |v_n(y)|^q + |w_n(y)|^q) dy < \frac{\eta}{3} \quad (3.6)$$

for every $A \subset Q$ with $|A| < \varepsilon$. By the q -equiintegrability of $\{w_n\}$ and $\{v_n\}$, and by Chebyshev's inequality there holds

$$|Q \cap (\{|w_n| > M\} \cup \{|v_n| > M\})| \leq \frac{1}{M^q} \int_Q (|w_n(y)|^q + |v_n(y)|^q) dy \leq \frac{C}{M^q}$$

for every $n \in \mathbb{N}$. Therefore, there exists M_0 satisfying

$$\sup_{n \in \mathbb{N}} |Q \cap (\{|w_n| > M_0\} \cup \{|v_n| > M_0\})| \leq \frac{\varepsilon}{2}. \quad (3.7)$$

By the uniform equicontinuity of the sequence $\{f_n(y, \cdot)\}$, there exists $\delta > 0$ such that, for every $\xi_1, \xi_2 \in \overline{B(0, M_0)}$, with $|\xi_1 - \xi_2| < \delta$, we have

$$\sup_{y \in Q} |f_n(y, \xi_1) - f_n(y, \xi_2)| < \varepsilon \quad (3.8)$$

for every $n \in \mathbb{N}$. By (3.5) and Egoroff's theorem, there exists a set $E_\varepsilon \subset Q$, $|E_\varepsilon| < \frac{\varepsilon}{2}$, such that

$$v_n \rightarrow 0 \quad \text{uniformly in } Q \setminus E_\varepsilon,$$

and, in particular,

$$|v_n(x)| < \delta \quad \text{for a.e. } x \in Q \setminus E_\varepsilon, \quad (3.9)$$

for every $n \geq n_0$, for some $n_0 \in \mathbb{N}$.

We observe that

$$\begin{aligned} \int_Q f_n(y, v_n(y) + w_n(y)) dy &= \int_{Q \cap \{|w_n| \leq M_0\} \cap \{|v_n| \leq M_0\}} f_n(y, v_n(y) + w_n(y)) dy \\ &\quad + \int_{Q \cap (\{|w_n| > M_0\} \cup \{|v_n| > M_0\})} f_n(y, v_n(y) + w_n(y)) dy. \end{aligned} \quad (3.10)$$

The first term in the right-hand side of (3.10) can be further decomposed as

$$\begin{aligned}
& \int_{Q \cap \{|w_n| \leq M_0\} \cap \{|v_n| \leq M_0\}} f_n(y, v_n(y) + w_n(y)) dy \\
&= \int_{(Q \setminus E_\varepsilon) \cap \{|w_n| \leq M_0\} \cap \{|v_n| \leq M_0\}} f_n(y, v_n(y) + w_n(y)) dy \\
&\quad + \int_{E_\varepsilon \cap \{|w_n| \leq M_0\} \cap \{|v_n| \leq M_0\}} f_n(y, v_n(y) + w_n(y)) dy \\
&= \int_{(Q \setminus E_\varepsilon) \cap \{|w_n| \leq M_0\} \cap \{|v_n| \leq M_0\}} f_n(y, w_n(y)) dy \\
&\quad + \int_{(Q \setminus E_\varepsilon) \cap \{|w_n| \leq M_0\} \cap \{|v_n| \leq M_0\}} (f_n(y, v_n(y) + w_n(y)) - f_n(y, w_n(y))) dy \\
&\quad + \int_{E_\varepsilon \cap \{|w_n| \leq M_0\} \cap \{|v_n| \leq M_0\}} f_n(y, v_n(y) + w_n(y)) dy \\
&= \int_Q f_n(y, w_n(y)) dy - \int_{E_\varepsilon \cap \{|w_n| \leq M_0\} \cap \{|v_n| \leq M_0\}} f_n(y, w_n(y)) dy \\
&\quad - \int_{Q \cap (\{|w_n| > M_0\} \cup \{|v_n| > M_0\})} f_n(y, w_n(y)) dy \\
&\quad + \int_{(Q \setminus E_\varepsilon) \cap \{|w_n| \leq M_0\} \cap \{|v_n| \leq M_0\}} (f_n(y, v_n(y) + w_n(y)) - f_n(y, w_n(y))) dy \\
&\quad + \int_{E_\varepsilon \cap \{|w_n| \leq M_0\} \cap \{|v_n| \leq M_0\}} f_n(y, v_n(y) + w_n(y)) dy.
\end{aligned}$$

We observe that by (3.7)

$$|E_\varepsilon \cup (\{|w_n| > M_0\} \cup \{|v_n| > M_0\})| < \varepsilon.$$

Hence, for $n \geq n_0$, by (3.4), (3.6), (3.8), and (3.9) we deduce the estimate

$$\begin{aligned}
& \left| \int_Q f_n(y, w_n(y)) dy - \int_Q f_n(y, v_n(y) + w_n(y)) dy \right| \\
& \leq \varepsilon + \int_{E_\varepsilon \cup (\{|w_n| > M_0\} \cup \{|v_n| > M_0\})} 2C(1 + |w_n(y)|^p + |v_n(y)|^p) dy \leq \varepsilon + \frac{2\eta}{3}.
\end{aligned} \tag{3.11}$$

The thesis follows by the arbitrariness of η . \square

We now prove our main result.

Proof of Theorem 1.1. The proof is subdivided into 4 steps. Steps 1 and 2 follow along the lines of [6, Proof of Theorem 1.1]. Step 3 is obtained by modifying [6, Lemma 3.5], whereas Step 4 follows by adapting an argument in [23, Proof of Theorem 1.2]. We only outline the main ideas of Steps 1 and 2 for convenience of the reader, whilst we provide more details for Steps 3 and 4.

Step 1:

The first step consists in showing that

$$\begin{aligned}
\mathcal{I}((u, v), D) = \inf \Big\{ & \liminf_{n \rightarrow +\infty} \int_D f(x, u(x), v_n(x)) dx : \{v_n\} \text{ is } q\text{-equiintegrable,} \\
& \mathcal{A}v_n \rightarrow 0 \text{ strongly in } W^{-1,s}(D; \mathbb{R}^l) \text{ for every } 1 < s < q \\
& \text{and } v_n \rightharpoonup v \text{ weakly in } L^q(D; \mathbb{R}^d) \Big\}.
\end{aligned}$$

This identification is proved by adapting [6, Proof of Lemma 3.1]. The only difference is the application of Lemma 3.1 instead of [6, Proposition 2.3 (i)].

Step 2:

The second step is the proof that $\mathcal{I}((u, v), \cdot)$ is the trace of a Radon measure absolutely continuous

with respect to $\mathcal{L}^N \llcorner \Omega$. This follows as a straightforward adaptation of [6, Lemma 3.4]. The only modifications are due to the fact that [6, Proposition 2.3 (i)] and [6, Lemma 3.1] are now replaced by Lemma 3.1 and Step 1.

Step 3:

We claim that

$$\frac{d\mathcal{I}((u, v), \cdot)}{d\mathcal{L}^N}(x_0) \geq Q_{\mathcal{A}(x_0)} f(x_0, u(x_0), v(x_0)) \quad \text{for a.e. } x_0 \in \Omega. \quad (3.12)$$

Indeed, since $g(x, \xi) := f(x, u(x), \xi)$ is a Carathéodory function, by Scorza-Dragoni Theorem there exists a sequence of compact sets $K_j \subset \Omega$ such that

$$|\Omega \setminus K_j| \leq \frac{1}{j}$$

and the restriction of g to $K_j \times \mathbb{R}^d$ is continuous. Hence, the set

$$\omega := \bigcup_{j=1}^{+\infty} (K_j \cap K_j^*) \cap \mathcal{L}(u, v), \quad (3.13)$$

where K_j^* is the set of Lebesgue point for the characteristic function of K_j and $\mathcal{L}(u, v)$ is the set of Lebesgue points of u and v , is such that

$$|\Omega \setminus \omega| \leq |\Omega \setminus K_j| \leq \frac{1}{j} \quad \text{for every } j,$$

and so $|\Omega \setminus \omega| = 0$. Let $x_0 \in \omega$ be such that

$$\lim_{r \rightarrow 0^+} \frac{1}{r^N} \int_{Q(x_0, r)} |u(x) - u(x_0)|^p dx = \lim_{r \rightarrow 0^+} \frac{1}{r^N} \int_{Q(x_0, r)} |v(x) - v(x_0)|^q dx = 0, \quad (3.14)$$

and

$$\frac{d\mathcal{I}((u, v), \cdot)}{d\mathcal{L}^N}(x_0) = \lim_{r \rightarrow 0^+} \frac{\mathcal{I}((u, v), Q(x_0, r))}{r^N} < +\infty, \quad (3.15)$$

where the sequence of radii r is such that $\mathcal{I}((u, v), \partial Q(x_0, r)) = 0$ for every r . (Such a choice of the sequence is possible due to Step 2).

By Step 1, for every r there exists a q -equiintegrable sequence $\{v_{n,r}\}$ such that

$$\begin{aligned} v_{n,r} &\rightharpoonup v \quad \text{weakly in } L^q(Q(x_0, r); \mathbb{R}^d), \\ \mathcal{A} v_{n,r} &\rightarrow 0 \quad \text{strongly in } W^{-1,s}(Q(x_0, r); \mathbb{R}^l) \text{ for every } 1 < s < q \end{aligned} \quad (3.16)$$

as $n \rightarrow +\infty$, and

$$\lim_{n \rightarrow +\infty} \int_{Q(x_0, r)} g(x, v_{n,r}(x)) dx \leq \mathcal{I}((u, v), Q(x_0, r)) + r^{N+1}.$$

A change of variables yields

$$\frac{d\mathcal{I}((u, v), \cdot)}{d\mathcal{L}^N}(x_0) \geq \liminf_{r \rightarrow 0^+} \lim_{n \rightarrow +\infty} \int_Q g(x_0 + ry, v(x_0) + w_{n,r}(y)) dy,$$

where

$$w_{n,r}(y) := v_{n,r}(x_0 + ry) - v(x_0) \quad \text{for a.e. } y \in Q.$$

Arguing as in [6, Proof of Lemma 3.5], Hölder's inequality and a change of variables imply

$$w_{n,r} \rightharpoonup 0 \quad \text{weakly in } L^q(Q; \mathbb{R}^d) \quad (3.17)$$

as $n \rightarrow +\infty$ and $r \rightarrow 0^+$, in this order. We claim that

$$\mathcal{A}(x_0 + r \cdot) w_{n,r} \rightarrow 0 \quad \text{strongly in } W^{-1,s}(Q; \mathbb{R}^l), \quad (3.18)$$

as $n \rightarrow +\infty$, for every r and every $1 < s < q$.

Indeed, let $\varphi \in W_0^{1,s'}(Q; \mathbb{R}^d)$. There holds

$$\begin{aligned} \langle \mathcal{A}(x_0 + r \cdot) w_{n,r}, \varphi \rangle_{W^{-1,s}(Q; \mathbb{R}^l), W_0^{1,s'}(Q; \mathbb{R}^l)} &= - \sum_{i=1}^N \left\{ r \int_Q \frac{\partial A^i(x_0 + ry)}{\partial x_i} v_{n,r}(x_0 + ry) \cdot \varphi(y) dy \right. \\ &\quad \left. + \int_Q A^i(x_0 + ry) v_{n,r}(x_0 + ry) \cdot \frac{\partial \varphi(y)}{\partial y_i} dy \right\} \\ &= - \sum_{i=1}^N \left\{ \frac{1}{r^{N-1}} \int_{Q(x_0, r)} \frac{\partial A^i(x)}{\partial x_i} v_{n,r}(x) \cdot \psi_r(x) dx + \frac{1}{r^{N-1}} \int_{Q(x_0, r)} A^i(x) v_{n,r}(x) \cdot \frac{\partial \psi_r(x)}{\partial x_i} dx \right\} \\ &= \frac{1}{r^{N-1}} \langle \mathcal{A} v_{n,r}, \psi_r \rangle_{W^{-1,s}(Q(x_0, r); \mathbb{R}^l), W_0^{1,s'}(Q(x_0, r); \mathbb{R}^l)}, \end{aligned}$$

where $\psi_r(x) := \varphi\left(\frac{x-x_0}{r}\right)$ for a.e. $x \in Q(x_0, r)$. Since $\psi_r \in W_0^{1,s'}(Q(x_0, r); \mathbb{R}^d)$ and

$$\|\psi_r\|_{W_0^{1,s'}(Q(x_0, r); \mathbb{R}^d)} \leq C(r) \|\varphi\|_{W_0^{1,s'}(Q; \mathbb{R}^d)},$$

we obtain the estimate

$$\|\mathcal{A}(x_0 + r \cdot) w_{n,r}\|_{W^{-1,s}(Q; \mathbb{R}^l)} \leq C(r) \|\mathcal{A} v_{n,r}\|_{W^{-1,s}(Q(x_0, r); \mathbb{R}^l)}.$$

Claim (3.18) follows by (3.16).

In view of (3.17) and (3.18), a diagonalization procedure yields a q -equiintegrable sequence $\{\hat{w}_k\} \subset L^q(Q; \mathbb{R}^d)$ satisfying

$$\hat{w}_k \rightharpoonup 0 \quad \text{weakly in } L^q(Q; \mathbb{R}^d), \quad (3.19)$$

$$\mathcal{A}(x_0 + r_k \cdot) \hat{w}_k \rightarrow 0 \quad \text{strongly in } W^{-1,s}(Q; \mathbb{R}^l) \quad \text{for every } 1 < s < q, \quad (3.20)$$

and

$$\frac{d\mathcal{I}((u, v), \cdot)}{d\mathcal{L}^N}(x_0) \geq \liminf_{k \rightarrow +\infty} \int_Q g(x_0 + r_k y, v(x_0) + \hat{w}_k(y)) dy. \quad (3.21)$$

For every $\varphi \in W_0^{1,s'}(Q; \mathbb{R}^l)$, $1 < s < q$, there holds

$$\begin{aligned} &\langle (\mathcal{A}(x_0 + r_k \cdot) - \mathcal{A}(x_0)) \hat{w}_k, \varphi \rangle_{W^{-1,s}(Q; \mathbb{R}^l), W_0^{1,s'}(Q; \mathbb{R}^l)} \\ &= - \sum_{i=1}^N \left[r_k \int_Q \frac{\partial A^i(x_0 + r_k y)}{\partial x_i} \hat{w}_k(y) \cdot \varphi(y) dy + \int_Q (A^i(x_0 + r_k y) - A^i(x_0)) \hat{w}_k(y) \cdot \frac{\partial \varphi(y)}{\partial y_i} dy \right]. \end{aligned}$$

Thus,

$$\|(\mathcal{A}(x_0 + r_k \cdot) - \mathcal{A}(x_0)) \hat{w}_k\|_{W^{-1,s}(Q; \mathbb{R}^l)} \leq r_k \sum_{i=1}^N \|A^i\|_{W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^l \times d)} \|\hat{w}_k\|_{L^q(Q; \mathbb{R}^d)}$$

for every $1 < s < q$. By (3.19) and (3.20) we conclude that

$$\mathcal{A}(x_0) \hat{w}_k \rightarrow 0 \quad \text{strongly in } W^{-1,s}(Q; \mathbb{R}^l) \quad \text{for every } 1 < s < q. \quad (3.22)$$

In view of (3.19) and (3.22), an adaptation of [6, Corollary 3.3] yields a q -equiintegrable sequence $\{w_k\}$ such that

$$\begin{aligned} w_k &\rightharpoonup 0 \quad \text{weakly in } L^q(Q; \mathbb{R}^d), \\ \int_Q w_k(y) dy &= 0 \quad \text{for every } k, \\ \mathcal{A}(x_0) w_k &= 0 \quad \text{for every } k, \end{aligned} \quad (3.23)$$

and

$$\liminf_{k \rightarrow +\infty} \int_Q g(x_0, v(x_0) + w_k(y)) dy \leq \liminf_{k \rightarrow +\infty} \int_Q g(x_0 + r_k y, v(x_0) + \hat{w}_k(y)) dy. \quad (3.24)$$

Finally, by combining (3.21), (3.23), and (3.24), and by the definition of \mathcal{A} -quasiconvex envelope for operators with constant coefficients, we obtain

$$\begin{aligned} \frac{d\mathcal{I}((u, v), \cdot)}{d\mathcal{L}^N}(x_0) &\geq \liminf_{k \rightarrow +\infty} \int_Q g(x_0, v(x_0) + w_k(y)) dy \\ &= \liminf_{k \rightarrow +\infty} \int_Q f(x_0, u(x_0), v(x_0) + w_k(y)) dy \geq Q_{\mathcal{A}(x_0)} f(x_0, u(x_0), v(x_0)) \end{aligned}$$

for a.e. $x_0 \in \Omega$. This concludes the proof of Claim (3.12).

Step 4:

To complete the proof of the theorem we need to show that

$$\frac{d\mathcal{I}((u, v), \cdot)}{d\mathcal{L}^N}(x_0) \leq Q_{\mathcal{A}(x_0)} f(x_0, u(x_0), v(x_0)) \quad \text{for a.e. } x_0 \in \Omega. \quad (3.25)$$

To this aim, let $\mu > 0$, and $x_0 \in \omega$ be such that (3.14) and (3.15) hold. Let $w \in C_{\text{per}}^\infty(\mathbb{R}^N; \mathbb{R}^d)$ be such that

$$\int_Q w(y) dy = 0, \quad \mathcal{A}(x_0)w = 0, \quad (3.26)$$

and

$$\int_Q f(x_0, u(x_0), v(x_0) + w(y)) dy \leq Q_{\mathcal{A}(x_0)} f(x_0, u(x_0), v(x_0)) + \mu. \quad (3.27)$$

Let $\eta \in C_c^\infty(\Omega; [0, 1])$ be such that $\eta \equiv 1$ in a neighborhood of x_0 and let r be small enough so that

$$Q(x_0, r) \subset \{x : \eta(x) = 1\} \quad \text{and} \quad Q(x_0, 2r) \subset \subset \Omega. \quad (3.28)$$

Consider a map $\varphi \in C_c^\infty(Q(x_0, r); [0, 1])$ satisfying

$$\mathcal{L}^N(Q(x_0, r) \cap \{\varphi \neq 1\}) < \mu r^N, \quad (3.29)$$

and define

$$z_m^r(x) := \varphi(x)w\left(\frac{m(x - x_0)}{r}\right) \quad \text{for } x \in \mathbb{R}^N. \quad (3.30)$$

We observe that $z_m^r \in L^q(\Omega; \mathbb{R}^d)$, and for $\psi \in L^{q'}(\Omega; \mathbb{R}^d)$ we have

$$\begin{aligned} \int_\Omega z_m^r(x) \cdot \psi(x) dx &= \int_\Omega \varphi(x)w\left(\frac{m(x - x_0)}{r}\right) \cdot \psi(x) dx \\ &= r^N \int_Q \varphi(x_0 + ry)w(my) \cdot \psi(x_0 + ry) dy. \end{aligned}$$

By (3.26) and by the Riemann-Lebesgue lemma we have

$$z_m^r \rightharpoonup 0 \quad \text{weakly in } L^q(\Omega; \mathbb{R}^d) \quad (3.31)$$

as $m \rightarrow +\infty$. We claim that

$$\limsup_{m \rightarrow +\infty} \|\mathcal{A}_\eta z_m^r\|_{W^{-1,q}(\Omega; \mathbb{R}^d)} \leq Cr^{\frac{N}{q}+1}, \quad (3.32)$$

where \mathcal{A}_η is the pseudo-differential operator defined in (2.2). Indeed, by (3.28) we obtain

$$\begin{aligned} \mathcal{A}_\eta z_m^r &= \mathcal{A} z_m^r - \mathcal{A}(x_0)z_m^r + \mathcal{A}(x_0)z_m^r \\ &= \sum_{i=1}^N \frac{\partial((A^i(x) - A^i(x_0))z_m^r(x))}{\partial x_i} + \sum_{i=1}^N A^i(x_0) \frac{\partial z_m^r(x)}{\partial x_i} - \sum_{i=1}^N \frac{\partial A^i(x)}{\partial x_i} z_m^r(x). \end{aligned} \quad (3.33)$$

By the regularity of the operators A^i and by a change of variables, the first term in the right-hand side of (3.33) is estimated as

$$\begin{aligned} & \left\| \sum_{i=1}^N \frac{\partial((A^i(x) - A^i(x_0))z_m^r(x))}{\partial x_i} \right\|_{W^{-1,q}(\Omega;\mathbb{R}^l)} \\ & \leq \sum_{i=1}^N \left\| (A^i(x) - A^i(x_0))\varphi(x)w\left(\frac{m(x-x_0)}{r}\right) \right\|_{L^q(Q(x_0,r);\mathbb{R}^l)} \\ & \leq \sum_{i=1}^N \|A^i\|_{W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^l \times d)} \|\varphi\|_{L^\infty(Q(x_0,r))} \|w(m\cdot)\|_{L^q(Q;\mathbb{R}^d)} r^{\frac{N}{q}+1} \leq C r^{\frac{N}{q}+1}. \end{aligned} \quad (3.34)$$

In view of (3.26) the second term in the right-hand side of (3.33) becomes

$$\sum_{i=1}^N A^i(x_0) \frac{\partial z_m^r(x)}{\partial x_i} = \sum_{i=1}^N A^i(x_0) \frac{\partial \varphi(x)}{\partial x_i} w\left(\frac{m(x-x_0)}{r}\right),$$

and thus converges to zero weakly in $L^q(\Omega;\mathbb{R}^l)$, as $m \rightarrow +\infty$, due to (3.26) and by the Riemann-Lebesgue lemma. Hence,

$$\left\| \sum_{i=1}^N A^i(x_0) \frac{\partial z_m^r(x)}{\partial x_i} \right\|_{W^{-1,q}(\Omega;\mathbb{R}^l)} \rightarrow 0 \quad \text{as } m \rightarrow +\infty \quad (3.35)$$

by the compact embedding of $L^q(\Omega;\mathbb{R}^l)$ into $W^{-1,q}(\Omega;\mathbb{R}^l)$. Finally, the third term in the right-hand side of (3.33) satisfies

$$\sum_{i=1}^N \frac{\partial A^i(x)}{\partial x_i} z_m^r(x) = \sum_{i=1}^N \frac{\partial A^i(x)}{\partial x_i} \varphi(x) w\left(\frac{m(x-x_0)}{r}\right),$$

which again converges to zero weakly in $L^q(\Omega;\mathbb{R}^l)$, as $m \rightarrow +\infty$, owing again to (3.26) and the Riemann-Lebesgue lemma. Therefore,

$$\left\| \sum_{i=1}^N \frac{\partial A^i(x)}{\partial x_i} z_m^r(x) \right\|_{W^{-1,q}(\Omega;\mathbb{R}^l)} \rightarrow 0 \quad \text{as } m \rightarrow +\infty. \quad (3.36)$$

Claim (3.32) follows by combining (3.34)–(3.36).

Consider the maps

$$v_m^r := P_\eta z_m^r,$$

where P_η is the projection operator introduced in (2.3). By Proposition 2.2 we have

$$\|v_m^r\|_{L^q(Q(x_0,r);\mathbb{R}^d)} \leq C \|z_m^r\|_{L^q(\Omega;\mathbb{R}^d)}, \quad (3.37)$$

$$\|v_m^r\|_{W^{-1,q}(Q(x_0,r);\mathbb{R}^d)} \leq C \|z_m^r\|_{W^{-1,q}(\Omega;\mathbb{R}^d)}, \quad (3.38)$$

$$\|\mathcal{A}_\eta v_m^r\|_{W^{-1,q}(Q(x_0,r);\mathbb{R}^l)} \leq C \|z_m^r\|_{W^{-1,q}(\Omega;\mathbb{R}^d)}, \quad (3.39)$$

$$\|v_m^r - z_m^r\|_{L^q(Q(x_0,r);\mathbb{R}^d)} \leq C (\|\mathcal{A}_\eta z_m^r\|_{W^{-1,q}(\Omega;\mathbb{R}^l)} + \|z_m^r\|_{W^{-1,q}(\Omega;\mathbb{R}^d)}). \quad (3.40)$$

By (3.31) and (3.37), the sequence $\{v_m^r\}$ is uniformly bounded in $L^q(Q(x_0,r);\mathbb{R}^d)$. Thus, there exists a map $v^r \in L^q(Q(x_0,r);\mathbb{R}^d)$ such that, up to the extraction of a (not relabelled) subsequence,

$$v_m^r \rightharpoonup v^r \quad \text{weakly in } L^q(Q(x_0,r);\mathbb{R}^d) \quad (3.41)$$

as $m \rightarrow +\infty$. Again by (3.31), and by the compact embedding of L^q into $W^{-1,q}$, we deduce that

$$z_m^r \rightarrow 0 \quad \text{strongly in } W^{-1,q}(\Omega;\mathbb{R}^d) \quad (3.42)$$

as $m \rightarrow +\infty$. Therefore, by combining (3.38) and (3.41), we conclude that

$$v_m^r \rightharpoonup 0 \quad \text{weakly in } L^q(Q(x_0,r);\mathbb{R}^d)$$

as $m \rightarrow +\infty$, and the convergence holds for the entire sequence. Additionally, by (3.28), (3.39), and (3.42), we obtain

$$\mathcal{A}v_m^r = \mathcal{A}_\eta v_m^r \rightarrow 0 \quad \text{strongly in } W^{-1,q}(Q(x_0, r); \mathbb{R}^l)$$

as $m \rightarrow +\infty$. Finally, by (3.32), (3.40), and (3.42), there holds

$$\lim_{r \rightarrow 0} \lim_{m \rightarrow +\infty} r^{-\frac{N}{q}} \|v_m^r - z_m^r\|_{L^q(Q(x_0, r); \mathbb{R}^d)} = 0. \quad (3.43)$$

We recall that, since x_0 satisfies (3.15), Step 1 yields

$$\frac{d\mathcal{I}(u, v)}{d\mathcal{L}^N}(x_0) = \lim_{r \rightarrow 0^+} \frac{\mathcal{I}((u, v); Q(x_0, r))}{r^N} \leq \liminf_{r \rightarrow 0^+} \liminf_{m \rightarrow +\infty} \frac{1}{r^N} \int_{Q(x_0, r)} f(x, u(x), v(x) + v_m^r(x)) dx. \quad (3.44)$$

We claim that

$$\frac{d\mathcal{I}(u, v)}{d\mathcal{L}^N}(x_0) = \lim_{r \rightarrow 0^+} \frac{\mathcal{I}((u, v); Q(x_0, r))}{r^N} \leq \liminf_{r \rightarrow 0^+} \liminf_{m \rightarrow +\infty} \frac{1}{r^N} \int_{Q(x_0, r)} g(x, v(x) + z_m^r(x)) dx, \quad (3.45)$$

where g is the function introduced in Step 3. Indeed, for every $r \in \mathbb{R}$, consider the function $g^r : Q \times \mathbb{R}^d \rightarrow [0, +\infty)$ defined as

$$g^r(y, \xi) := g(x_0 + ry, \xi) \quad \text{for every } y \in Q, \xi \in \mathbb{R}^d.$$

Since $x_0 \in \omega$, by (3.13) there exists K_j such that $x_0 \in K_j$. In particular, this yields the existence of $r_0 > 0$ such that for $r \leq r_0$, the maps g^r are continuous on $Q \times \mathbb{R}^d$, and the family $\{g^r(y, \cdot)\}$ is equicontinuous in \mathbb{R}^d , uniformly with respect to y . A change of variables yields

$$\begin{aligned} & \frac{1}{r^N} \left| \int_{Q(x_0, r)} f(x, u(x), v(x) + v_m^r(x)) dx - \int_{Q(x_0, r)} f(x, u(x), v(x) + z_m^r(x)) dx \right| \\ &= \left| \int_Q g^r(y, v(x_0 + ry) + v_m^r(x_0 + ry)) dy - \int_Q g^r(y, v(x_0 + ry) + z_m^r(x_0 + ry)) dy \right|. \end{aligned}$$

On the other hand, by (3.43) we have

$$\lim_{r \rightarrow 0} \lim_{m \rightarrow +\infty} \|z_m^r(x_0 + r \cdot) - v_m^r(x_0 + r \cdot)\|_{L^q(Q; \mathbb{R}^d)} = \lim_{r \rightarrow 0} \lim_{m \rightarrow +\infty} r^{-\frac{N}{q}} \|z_m^r - v_m^r\|_{L^q(Q(x_0, r); \mathbb{R}^d)} = 0.$$

Therefore, by a diagonal procedure we extract a subsequence $\{m_r\}$ such that

$$\begin{aligned} & \limsup_{r \rightarrow 0} \limsup_{m \rightarrow +\infty} \left| \int_Q g^r(y, v(x_0 + ry) + v_m^r(x_0 + ry)) dy - \int_Q g^r(y, v(x_0 + ry) + z_m^r(x_0 + ry)) dy \right| \\ &= \lim_{r \rightarrow 0} \left| \int_Q g^r(y, v(x_0 + ry) + v_{m_r}^r(x_0 + ry)) dy - \int_Q g^r(y, v(x_0 + ry) + z_{m_r}^r(x_0 + ry)) dy \right|, \end{aligned} \quad (3.46)$$

and

$$z_{m_r}^r(x_0 + r \cdot) - v_{m_r}^r(x_0 + r \cdot) \rightarrow 0 \quad \text{strongly in } L^q(Q; \mathbb{R}^d).$$

In view of (3.14), (3.30) and the Riemann-Lebesgue lemma, the sequence $\{v(x_0 + r \cdot) + z_{m_r}^r(x_0 + r \cdot)\}$ is q -equiintegrable in Q . Hence, by (H) we are under the assumptions of Proposition 3.2, and we conclude that

$$\lim_{r \rightarrow 0} \left| \int_Q g^r(y, v(x_0 + ry) + v_{m_r}^r(x_0 + ry)) dy - \int_Q g^r(y, v(x_0 + ry) + z_{m_r}^r(x_0 + ry)) dy \right| = 0. \quad (3.47)$$

Claim (3.45) follows by combining (3.46) with (3.47).

Arguing as in [6, Proof of Lemma 3.5], for every $x_0 \in \omega$ (where ω is the set defined in (3.13)) we have

$$\begin{aligned} & \liminf_{r \rightarrow 0^+} \liminf_{m \rightarrow +\infty} \frac{1}{r^N} \int_{Q(x_0, r)} f(x, u(x), v(x) + z_m^r(x)) dx \\ & \leq \liminf_{r \rightarrow 0^+} \liminf_{m \rightarrow +\infty} \frac{1}{r^N} \int_{Q(x_0, r)} f(x_0, u(x_0), v(x_0) + z_m^r(x)) dx, \end{aligned}$$

hence by (3.45) we deduce that

$$\frac{d\mathcal{I}(u, v)}{d\mathcal{L}^N}(x_0) \leq \liminf_{r \rightarrow 0^+} \liminf_{m \rightarrow +\infty} \frac{1}{r^N} \int_{Q(x_0, r)} f(x_0, u(x_0), v(x_0) + z_m^r(x)) dx.$$

By (3.30) we obtain

$$\begin{aligned} \frac{d\mathcal{I}(u, v)}{d\mathcal{L}^N}(x_0) & \leq \liminf_{r \rightarrow 0^+} \liminf_{m \rightarrow +\infty} \frac{1}{r^N} \int_{Q(x_0, r)} f(x_0, u(x_0), v(x_0) + z_m^r(x)) dx \\ & \leq \liminf_{r \rightarrow 0^+} \liminf_{m \rightarrow +\infty} \frac{1}{r^N} \left\{ \int_{Q(x_0, r)} f\left(x_0, u(x_0), v(x_0) + w\left(\frac{m(x - x_0)}{r}\right)\right) dx \right. \\ & \quad \left. + \int_{Q(x_0, r) \cap \{\varphi \neq 1\}} f\left(x_0, u(x_0), v(x_0) + \varphi(x)w\left(\frac{m(x - x_0)}{r}\right)\right) dx \right\}. \end{aligned}$$

The growth assumption (H) and estimate (3.29) yield

$$\begin{aligned} & \int_{Q(x_0, r) \cap \{\varphi \neq 1\}} f\left(x_0, u(x_0), v(x_0) + \varphi(x)w\left(\frac{m(x - x_0)}{r}\right)\right) dx \\ & \leq C \int_{Q(x_0, r) \cap \{\varphi \neq 1\}} \left(1 + \left|w\left(\frac{m(x - x_0)}{r}\right)\right|^q\right) dx \\ & \leq C(1 + \|w\|_{L^\infty(\mathbb{R}^N; \mathbb{R}^d)}^q) \mathcal{L}^N(Q(x_0, r) \cap \{\varphi \neq 1\}) \leq C\mu r^N. \end{aligned} \tag{3.48}$$

Thus, by (3.48), the periodicity of w , and Riemann-Lebesgue lemma, we deduce

$$\begin{aligned} \frac{d\mathcal{I}(u, v)}{d\mathcal{L}^N}(x_0) & \leq C\mu + \liminf_{r \rightarrow 0^+} \liminf_{m \rightarrow +\infty} \frac{1}{r^N} \int_{Q(x_0, r)} f\left(x_0, u(x_0), v(x_0) + w\left(\frac{m(x - x_0)}{r}\right)\right) dx \\ & = C\mu + \liminf_{m \rightarrow +\infty} \int_Q f(x_0, u(x_0), v(x_0) + w(my)) dy \\ & = C\mu + \int_Q f(x_0, u(x_0), v(x_0) + w(y)) dy \\ & \leq C\mu + Q_{\mathcal{A}(x_0)} f(x_0, u(x_0), v(x_0)), \end{aligned}$$

where the last inequality is due to (3.27). Letting $\mu \rightarrow 0^+$ we conclude (3.25). \square

ACKNOWLEDGEMENTS

The authors thank the Center for Nonlinear Analysis (NSF Grant No. DMS-0635983), where this research was carried out, and also acknowledge support of the National Science Foundation under the PIRE Grant No. OISE-0967140. The research of I. Fonseca and E. Davoli was funded by the National Science Foundation under Grant No. DMS- 0905778. E. Davoli acknowledges the support of the Austrian Science Fund (FWF) projects P 27052 and I 2375. The research of I. Fonseca was further partially supported by the National Science Foundation under Grant No. DMS-1411646.

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